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Relativistic quantum statistical properties of many-fermion system in a mesoscopic ring

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Abstract. The relativistic quantum statistical properties of a charged many-fermion system with single fermion rest mass m_0 confined in a mesoscopic ring are investigated. The small ring is threaded by an Aharonov–Bohm magnetic flux ϕ. We have performed analytical calculations starting from evaluating the logarithm of the grand partition functions by using the Mellin transform. The persistent currents, number densities, total energies, heat capacity and hence the fluctuations of the total energies are obtained both in the weakly degenerate and degenerate situations for finite temperature. These thermodynamic functions oscillate periodically in ϕ with period $\phi_0 = h/q$, and they are related to the modified Bessel functions of the second kind $K_\nu(r)$. For the weakly degenerate case, where the chemical potential $\mu \leqslant m_0$, we find they decay exponentially with respect to the decrease in temperature, and their magnitudes also decrease in the form of modified exponential functions as the circumference L increases. In the degenerate case, where $\mu > m_0$, these functions behave complicatedly in the form of generalized hypergeometric functions, which decay in the power series forms at very low temperature. The non-relativistic limits are given by considering the asymptotic behaviours of $K_\nu(r)$.

1. Introduction

Between the measures of macroscopic and microscopic systems there exist so-called mesoscopic systems [1], whose measures lie from millimicron to micron, and in which about 10^{11} microparticles are contained. The physical quantities of mesoscopic systems are also the statistical averages of a large number of microscopic particles. In contrast to macroscopic physics, the dimensions being smaller than the phase coherent lengths, the microparticles keep their phase memories due to the fact that only elastic scattering procedure takes place between them [2]. The superpositions of particle wavefunctions with coherent phase exhibit non-local properties and strong fluctuations. Therefore, a great number of curious phenomena and profound physical connotations are contained in the mesoscopic frontier. Large ensemble theories should be amended and supplemented in the small structures. Because the surface effect of mesoscopic systems cannot be neglected, physical properties rely on the forms of objects, and hence we will also encounter many difficulties during our investigation.

In 1957, Landauer proposed that the resistance of a one-dimensional mesoscopic system can be calculated by barrier penetrating model, and he obtained a formula expressed in terms of the transmission coefficient [3]. During the transport in the disordered medium,

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the electrons undergo elastic scattering of impurity atoms, holes and crystal boundaries, etc. Landauer–Büttiker transport theory studies the overall transport characteristic of a simplified system connected to different terminals [4]. The two-dimensional system having no external interaction is conncected to terminal poles with chemical potentials μ_i $(i = 1, \ldots, N)$ by complete wires. The electrons emitted from the reservoirs lose their phase memories completely when entering the terminal poles because they experience inelastic scattering. Büttiker's formula very straightforwardly associates the conductance measured from the terminals with electron scattering matrix of the system. The theory successfully explains the conductance fluctuation [5], ballistic curve transport [6], classical and quantum Hall effect [7], etc, of mesoscopic quantum interferential transport.

The first experiment to show the superposition of coherent electron waves affecting transport properties was performed by Sharvin *et al* [8] in 1981 by investigating the magnetic resistance of a cylindrical film of magnesium. Then Altshuler *et al* performed an experiment to examine a cylindrical film of lithium. All these showed that the magnetic resistance oscillates with a period $\Phi_0/2$, where $\Phi_0 = h/e$; the magnetic resistance of mesoscopic system is smaller than classical one; in the absence of external magnetic field the magnetic resistance reaches its maximum value [9]. The observation performed by Webb *et al* on the normal gold ring with millimicron size exhibits the Aharonov–Bohm effect explicitly [10]. Stone simulated the structure of magnetic resistance oscillation numerically by using a computer, and he predicted that the second quantum contribution to the magnetic resistance of a gold ring is the oscillation of magnetic resistance with period Φ_0 superposed on the background of random fluctuating resistance [11]. Altshuler *et al* studied the weakly local conductance by calculating the correlation function, and they obtained the conductance correction of conjugate electron waves [12]. Altshuler *et al* investigated the mesoscopic conductance G and density fluctuations of Anderson model by a field-theoretical approach. They pointed out that at not too large deviations from the mean value, the fluctuations are described within the framework of one-parameter scaling. In the metallic region $(G \gg e^2/h)$ the one-parameter scaling leads to fluctuations that deviate greatly from a Gaussian [13]. Tang *et al* discovered that the conductances of mesoscopic systems are sensitive functions of DC voltage by using scattering theory [14]. Lee *et al* employed the technique of correlation function to calculate the mean square fluctuation, and they found that $G \propto e/\Phi_0$ [15]. For any sample, so long as its dimension is smaller than the phase coherence correlation length, the variation with magnetic field of the conductance fluctuation is universal. The fluctuation amplitude is about e/Φ_0, depends weakly on the form of the samples and does not relate to the mean resistance. The pictures of magnetic resistance fluctuations are different, even though they are measured under the same conditions. This is due to the fact that the number of scattering centres and distributions are different in the same sample.

Cheung *et al* performed analytical calculations and computer simulations to study the persistent current in a small isolated one-dimensional normal metal rings with circumference L enclosing a magnetic flux. Scattered by a random potential, the electric current vibrates with period Φ_0, and it decreases exponentially with temperature T. Under a certain averaging procedure, the vibrating period changes to $\Phi_0/2$ [16]. Ambegaokar *et al* calculated the equilibrium current in a mesoscopic normal metal ring to first order in the screened electron–electron interaction. They discovered that the average current in an ensemble of such a ring is periodic in the flux with period $\Phi_0/2$, and at zero temperature it is to order of magnitude $I \sim ev_F\ell/L^2$, where ℓ is the mean free path. The effect of current decreasing exponentially with temperature is determined on a scale of the coherent energy hD/L^2, where D is the electronic diffusion constant [17]. Efectov calculated a dynamic current in a mesoscopic ring subjected to a time-dependent external magnetic field by using linear-response theory and the supersymmetric method. He showed that the current remains finite in the limit of the frequency approaching zero, and that the excess currents can be obtained from thermodynamic considerations [18]. Nathanson *et al* studied the effect of a magnetic flux threading a perfect mesoscopic one-dimensional ring upon Peierls instability. Within the mean-field approximation, the Peierls transition temperature oscilliates with the flux. Fluctuations due to the finiteness of the ring destroy this effect as they smear the phase transition. When the effect of the Peierls instability upon the oscillatory behaviour of thermodynamic quantities of the ring with the flux is considered, it is found that the amplitude of oscillation is suppressed significantly even for a very small ring [19].

D'Amato and Pastawki proposed a model in which they employed perfect lateral leads to couple a one-dimensional chain. The coupling introduces damping of the wavefunction by adding an imaginary part to the self-energy [20]. Datta *et al* presented a form of Hamiltonian assuming that inelastic scattering is caused by a distribution of independent oscillators. Each oscillator interacts with the electrons through a δ-potential in space. They also obtained a steady-state quantum transport equation from the Dyson equation [21]. Chen and Sorbello investigated a one-dimensional mesoscopic system coupled with electron reservoirs. They considered both the elastic and inelastic scattering to obtain a formula for the conductance. They started from the ensemble density matrices of electrons and impurity to find explicit expressions for the effective temperature of the non-equilibrium electrons and for the relaxation rate of the impurity towards the thermally exited steady state. In the case of high bias and high lattice temperature, the impurity follows a Boltzmann distribution with a time-dependent temperature [22].

There is much interest in the study of transport in mesoscopic systems with strong electron–electron correlation [23–25]. It is believed that these systems constitute a new area of research where novel phenomena associated with quantum coherence in the presence of electron–electron interaction can be probed. Ng investigated the problem of nonlinear resonant tunneling through an Anderson impurity. He found that to order $1/N$, the Kondo resonance is not destroyed by any finite potential difference between external poles [26]. Heinonen and Johnson presented an approach to the study of steady-state mesoscopic transport based on the maximum entropy principle formulation of non-equilibrium statistical mechanics [27]. This approach is not limited to the linear response regime, and it yields the quantization observed in the integer quantum Hall effect at large currents. Resistance fluctuations in multiple-lead geometries, random-matrix theory of mesoscopic fluctuations, the current–current correlation functions, dynamics and thermodynamics of metallic rings are discussed in detail [28–31].

Small metallic rings provide excellent objects for studying mesoscopic physics both theoretically and experimentally. Persistent currents are the generally attentive problems which occur not only in isolated rings but also in rings connected via leads to electron reservoirs [32–34]. The persistent currents referring to the magnetic flux threading through the rings provide an important clue to understanding mesoscopic physics. Statistical mechanics is a natural approach for research into the problems of thermodynamics for a mesoscopic sample from which one can extrapolate complex behaviour of the system by thermodynamic functions. The main previous investigations on mesoscopic systems are based on non-relativistic point of view. However, it is necessary to investigate the small sample by relativistic statistical mechanics. The dynamical problem of interacting particles is rather difficult, however. If interactions can be ignored, the effects of relativistic kinematics on the mesoscopic properties of a many-body system can be studied in terms of free-particle models.

The purpose of this paper is to examine a mesoscopic ring threading an Aharonov–Bohm magnetic flux, in which N charged ideal fermions are contained. In section 2, we present some useful formulae of thermodynamics for calculating macroscopic observables. From the Mellin transform we give the general formulation of the logarithm of the grand partition function of this free-fermion system. Sections 3 and 4 are devoted to explicit calculations in the cases of weak degeneration and degeneration, respectively. The persistent currents, total energies, number densities, and the ensemble fluctuation of the energy are obtained in these sections. Section 5 contains a brief discussion.

2. Relativistic ensemble formulation

Consider a system of N charged fermions with rest mass m_0, charge q, and spin s for each fermion confined in a circular ring with circumference L. Each fermion is dominated by the Dirac equation. Because particle–particle interactions are strictly excluded, no negative energy states are excited, and the question of pair-production never arises. These fermions submit to Fermi–Dirac statistics, and hence to Pauli's exclusion principle. If all negative energy levels are filled, and all the positive energy levels are empty, this state is called a vacuum state. Thus, the physical vacuum which has the lowest energy is obtained by filling the Dirac sea. The value of the vacuum state energy cannot be observed. We assume that the magnetic flux ϕ threads the ring axially, but there is no magnetic field on the ring. Therefore the fermions always move in a field-free space. In the one-channel case the spatial degree of freedom of single particle is the azimuthal angle θ, the vector potential $\mathbf{A}$ may be chosen to have the form $\boldsymbol{A} = 2\pi\hat{\theta}\phi/L^2$, where $\hat{\theta}$ is the unit vector in the direction of azimuthal angle θ [16]. Instead of using θ as the spatial degree of freedom of a single particle, we employ the spatial variable $x = L\theta/2\pi$. The Dirac equation of single fermion in the vector potential $\boldsymbol{A}$ is

$$[\gamma_\mu(\partial_\mu - \mathrm{i}(q/\hbar)A_\mu(x)) + m_0/\hbar]\psi(x) = 0 \tag{1}$$

where γ_μ are Dirac matrices, $A_\mu = (\boldsymbol{A}, \mathrm{i}\varphi)$, $(\mu = 1, 2, 3, 4)$, and we have set the velocity of light C to be unit, i.e. $C = 1$. Here φ is the scalar potential. As a non-relativistic Schrödinger equation, the Dirac equation (1) is gauge invariant by making the second kind of gauge transformation for potential $A_\mu(x)$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x) \tag{2}$$

and by the phase factor transformation for the wavefunction $\psi(x)$

$$\psi(x) \to \psi'(x) = \exp[\mathrm{i}q\Lambda(x)/\hbar]\psi(x)\,. \tag{3}$$

For our system, it can be disposed by considering the situation where the fermions move independently of the magnetic field, but are affected by the vector potential. The Hamiltonian transfers to the free particle form; however, the wavefunction is weighted by a phase factor. As discussed in [35], we deal with the circumstance where the field does not appear explicitly in the Hamiltonian, but the boundary conditions are modified:

$$\psi(L) = \exp\left(\mathrm{i}\frac{2\pi\phi}{\phi_0}\right)\psi(0) \qquad \frac{\mathrm{d}\psi(L)}{\mathrm{d}x} = \exp\left(\mathrm{i}\frac{2\pi\phi}{\phi_0}\right)\frac{\mathrm{d}\psi(0)}{\mathrm{d}x} \tag{4}$$

where $\phi_0 = h/q$, and h is Planck's constant, so that each fermion in this system has the discrete energy spectrum

$$E(k_n) = (\hbar^2k_n^2 + m_0^2)^{1/2} \tag{5}$$

where k_n is the dimensionless magnitude of the quantized propagation vector

$$k_n = \frac{2\pi}{L}\left(n + \frac{\phi}{\phi_0}\right) \qquad n = 0, \pm 1, \pm 2, \dots \,. \tag{6}$$

Quantum mechanics gives a correct description of the microscopic behaviour of the constituent particles of the N-fermion system, and the quantum numbers describing the possible states take on discrete values in the closed mesoscopic ring. Each member of the discrete set of energy E_n is an implicit function of numerous parameters, such as the circumference L, magnetic flux ϕ, and chemical potential. To find the average values of the finite system observables, we should investigate the quantum relativistic grand canonical ensemble.

A completely general expression for the grand canonical partition function describing an N-particle system in thermal equilibrium is defined by

$$Z_{\mathrm{G}}(\beta, \phi) = \mathrm{Tr}\exp[-\beta(\hat{H} - \mu\hat{N})] \tag{7}$$

where $\hat{H}$ is the Hamiltonian of the system, is $\hat{N} = \sum_i \hat{n}_i$ the total number operator, and $\hat{n}_i$ is the operator such that its expectation value at any instant gives the number of fermions in state i. In equation (7), β is the inverse of the temperature T, i.e. $\beta = 1/K_{\mathrm{B}}T$, μ the chemical potential of fermion, with K_{B} Boltzmann's constant. Evaluation of the traces in (7) is a rather formidable task in general, major complications arising from the interactions among particles in the system as well as from all the effects of any external fields. In future work we will study the mesoscopic systems for incorporating these particle interactions explicitly in the calculations, but the present paper is concerned primarily with situations of neglecting the interactions. In this grand cononical emsemble the independent thermodynamical parameters are chosen to be L, T and μ.

Due to the fact that the system is subject to Pauli's principle, the independent global states of the fermions are specified by explicitly invoking this principle in the form of a constraint limiting the values of the occupation numbers to $n_i = 0, 1$. The grand partition function of the model is then formulated as the product of the discrete functions for each single-particle state [36]:

$$Z_{\mathrm{G}}(\beta, \phi) = \prod_{i=-\infty}^{\infty} [1 + \exp(-\beta E(k_i) - \alpha)] \qquad \alpha = -\beta\mu \,. \tag{8}$$

The magnitude of the chemical potential μ in general takes values in the range $-\infty < \mu < \infty$. The free energy $F(\beta, \phi)$ can immediately be determined according to

$$F(\beta, \phi) = -K_{\mathrm{B}}T \ln Z_{\mathrm{G}}(\beta, \phi) \,. \tag{9}$$

The relevant thermodynamic functions can be derived from equation (8) by direct differentiation. The number density of the system can be found from

$$n = \frac{1}{\beta L}\frac{\partial}{\partial\mu} \ln Z_{\mathrm{G}}(\beta, \phi) \tag{10}$$

and the total energy is then

$$E(\beta, \phi) = \mu N - \frac{\partial}{\partial\beta} \ln Z_{\mathrm{G}}(\beta, \phi) \tag{11}$$

where $N = nL$ is the total number of fermions.

Of considerable interest to the experimentalist is the persistent current which is a measurable quantity that describes the response of the system to external magnetic flux stimuli. At finite temperatures, one can calculate the current from the free energy shown in (9) originated by Yang and Byers [35]:

$$I = -\frac{\partial F}{\partial \phi}. \tag{12}$$

If the system obeys Maxwell's relations, the heat capacity is defined by the partial derivative of energy with respect to temperature at constant length L:

$$C_L = \left(\frac{\partial E}{\partial T}\right)_L . \tag{13}$$

The measuring pressure can be expressed explicitly as

$$p = \frac{1}{\beta L} \ln Z_{\mathrm{G}}(\beta, \phi) . \tag{14}$$

Since the partition function (8) is uniquely determined by the energy levels which vary periodically with period ϕ_0, the partition function, free energy and other thermodynamic functions of the system are obviously periodic functions of ϕ with period ϕ_0. Hence, all the measured mesoscopic quantities are periodic functions of ϕ. Furthermore, the system under consideration is invariant under time reversal, and the free energy remains unchanged if the system rotates around the origin; thereby, ϕ changes its sign. Thus, $F(\beta, \phi)$ must be an even function of ϕ, and the current derived from (12) must be an odd function of ϕ.

Now we employ a very powerful and useful recipe developed in terms of the inverse Mellin transform by Grandy *et al* [37] to evaluate $\ln Z_{\mathrm{G}}$ in arbitrary temperature and chemical potential. Under appropriate conditions on the function $g(t)$, the inverse Mellin transform reads

$$\tilde{g}(x) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} g(t) x^{-t} \, \mathrm{d}t . \tag{15}$$

From standard sources one can look up the transform pair

$$\ln|1 + x| \qquad \pi t^{-1} \csc(\pi t) \qquad -1 < \mathrm{Re}\, t < 0.$$

The integral and sum can be interchanged assuming both of them are convergent uniformly, hence the logarithm of grand partition function (8) for free relativistic fermions yields

$$\ln Z_{\mathrm{G}}(\beta, \phi) = \frac{1}{2\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \frac{1}{t \sin(\pi t)} \exp(\beta \mu t) Z_{\mathrm{B}}(t\beta, \phi) \, \mathrm{d}t \qquad 0 < a < 1 \tag{16}$$

where $Z_{\mathrm{B}}(t\beta, \phi)$ is the single-particle canonical partition function in Boltzmann statistics

$$Z_{\mathrm{B}}(t\beta, \phi) = \sum_i \exp(-t\beta E_i(\phi)) \tag{17}$$

with the summation running over all single-particle states.

Utilizing the basic Possion summation formula for a given function $f(x)$:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \exp(\mathrm{i}2\pi \ell x) \, \mathrm{d}x \tag{18}$$

we find that equation (17) in this system explicitly is as follows:

$$Z_{\mathrm{B}}(t\beta, \phi) = \frac{\xi \eta}{\pi} \left[K_1(zt) + \sum_{\ell=1}^{\infty} \sum_{\nu=0}^{\infty} P_{\nu\ell}(tz) K_{\nu+1}(zt) \cos\left(2\pi \ell \frac{\phi}{\phi_0}\right) \right] \tag{19}$$

where

$$P_{\nu\ell}(tz) = 2(-1)^{\nu}\frac{1}{\nu!}\left[\frac{(\xi\ell)^2}{2zt}\right]^{\nu} \qquad \xi = Lm_0/\hbar \qquad z = \beta m_0 \qquad \eta = 2s+1\,.$$

Here $K_{\nu}(r)$ are the modified Bessel functions of the second kind [38]. In equation (19), because the subscript of the Bessel functions is positive integer, $K_{\nu+1}(r)$ are expressed by ascending series with respect to r for each integer ν.

In equation (16), the contour of integration can usually be closed with a semicircle, and the integral can be evaluated by means of Cauchy's residue theorem. For $\nu > -1$, and $\mathrm{Re}\, zt > 0$, it follows that $K_{\nu}(zt) > 0$, and is a regular function of zt throughout the complex plane cut along the negative real axis, with a branch point at the origin. Therefore, it is obvious that the integrand has a logarithmic branch point at the origin, and simple poles exist at $t = m, m = 0, \pm 1, \pm 2, \ldots$. The logarithm of $Z_{\mathrm{G}}(\beta, \phi)$ in (16) describes the quantum properties of free fermions in the mesoscopic ring over the entire temperature range and for $-\infty < \mu < +\infty$. Once the function in (16) is evaluated, we can find various expectation values of observable quantities, such as number density, total energy, persistent current, etc, from the corresponding equations. In the following two sections we consider the weakly degenerate and degenerate circumstances for the caculations.

3. The weakly degenerate case

3.1. General formulae

In this section we consider the weakly degenerate case, where $\mu \leqslant m_0$. The integral in (16) is evaluated for this case by closing the contour to the right with a semicircle of radius R, because the integrand in the formula vanishes on the semicircle as $R \to \infty$. Now the contour encloses only the poles at the positive integers $m = 1, 2, \ldots$. The negative simple poles have no contributions to the integral. The integration is carried out over the contour traversing in the clockwise direction. Hence, by Cauchy's theorem the integral completely around the contour is just that $-2\pi\mathrm{i}$ times the sum of the residues at simple poles. Actually equation (16) reduces to

$$\ln Z_{\mathrm{G}}(\beta, \phi) = \sum_{m=1}^{\infty}(-1)^{m+1}\frac{1}{m}\exp(m\beta\mu)Z_{\mathrm{B}}(m\beta, \phi)\,. \tag{20}$$

Substituting this result into (9) one obtains the free energy of this system in the weakly degenerate case. The persistent current at finite temperature thus yields directly from (12)

$$I(\beta, \phi) = B_0\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\nu=0}^{\infty}(-1)^m\frac{\ell}{m}P_{\nu\ell}(mz)\exp(m\beta\mu)K_{\nu+1}(mz)\sin\left(2\pi\ell\frac{\phi}{\phi_0}\right) \tag{21}$$

where

$$B_0 = \frac{q\xi\eta}{\beta\hbar\pi}\,.$$

This is the relativistic persistent current formula in the weakly degenerate, and it can be derived from another approach instead of the inverse Mellin transform. After taking logarithms in (8), we have

$$\ln Z_{\mathrm{G}}(\beta, \phi) = \sum_{n=-\infty}^{\infty}\ln[1+\exp(-\beta E(k_n)-\alpha)] \tag{22}$$

where the summation runs over all single-fermion states. Employing the Possion summation formula (18) and equations (9), (12) we obtain

$$I(\beta,\phi) = \sum_{\ell=-\infty}^{\infty} I_\ell(\beta,\phi) \tag{23}$$

where

$$I_\ell(\beta,\phi) = -\eta \int_{-\infty}^{\infty} \frac{\partial E(x)}{\partial \phi} f(E) \exp(\mathrm{i}2\pi\ell x)\, \mathrm{d}x \tag{24}$$

and

$$\frac{\partial E}{\partial \phi} = \frac{qk\overline{h}}{LE} \qquad f(E) = [exp(\beta E + \alpha) + 1]^{-1}\,.$$

Here E is the single-fermion energy level. Substituting

$$k = \frac{2\pi}{L}\left(x + \frac{\phi}{\phi_0}\right)$$

into (24), one finds

$$I(\beta,\phi) = -\frac{2\overline{h}q\eta}{\pi} \sum_{\ell=1}^{\infty} \sin\left(2\pi\ell\frac{\phi}{\phi_0}\right) \int_0^{\infty} \frac{k}{E(k)} f(k) \sin(Lk\ell)\, \mathrm{d}k \tag{25}$$

or equivalently

$$I(\beta,\phi) = -\frac{2q\eta}{\overline{h}\pi} \sum_{\ell=1}^{\infty} \sin\left(2\pi\ell\frac{\phi}{\phi_0}\right) \int_{m_0}^{\infty} f(E) \sin[L\ell(E^2 - m_0^2)^{1/2}/\overline{h}]\, \mathrm{d}E\,. \tag{26}$$

For the weakly degenerate situation where $\mu \leqslant m_0$, the Fermi distribution function can be expanded to the series

$$f(E) = \sum_{m=1}^{\infty} (-1)^{m+1} \exp[-m(\beta E + \alpha)]\,. \tag{27}$$

Expanding the sine function in the integrand and substituting (27) into (25), we arrive at the same formula (21). The current obtained here is available for the weakly degenerate case where the fugacity expansions are used.

The persistent current (21) is an equilibrium property of the ring requiring that the phase-coherence length be of order L. The free energy refers to the thermal equilibrium of the system at a fixed value of magnetic flux, however it retains its significance for the magnetic flux varying slowly to permit at any instant the establishment of equilibrium. From equation (21) we see that the current is an odd function of the magnetic flux, and it varies periodically with the flux. In the ideal ring without impurity, the current is a function of the number of fermions, chemical potential and temperature. It is highly sample sensitive, decaying exponentially with decreasing temperature, and the magnitude fluctuates around its non-relativistic value.

The number density is determined directly from equations (10) and (20) to give

$$n = \frac{1}{L} \sum_{m=1}^{\infty} (-1)^{m+1} \exp(m\beta\mu) Z_{\mathrm{B}}(m\beta,\phi)\,. \tag{28}$$

The number density (28) is a positive function of the chemical potential, temperature, and magnetic flux. It varies periodically with respect to ϕ as a cosine function with period ϕ_0, and it decays rapidly with decreasing temperature, as well as increasing circumference. This gives the relativistic correction of the usual non-relativistic mesoscopic system. It fluctuates violently with the interference of wavefunctions.

The total energy can be found by noting the derivative of the modified Bessel function with respect to β:

$$\frac{\partial}{\partial\beta} K_\nu(b\beta) = \frac{\nu}{\beta} K_\nu(b\beta) - bK_{\nu+1}(b\beta) .$$

From equation (11) and the above derivative formula one has

$$E(\beta, \phi) = \frac{\xi\eta}{\pi} \sum_{m=1}^{\infty} (-1)^{m+1} \exp(m\beta\mu)\tilde{L}(mz) \tag{29}$$

where

$$\tilde{L}(mz) = L_0(mz) + \sum_{\ell=1}^{\infty}\sum_{\nu=0}^{\infty} P_{\nu\ell}(mz)L_\nu(mz) \cos\left(2\pi\ell\frac{\phi}{\phi_0}\right)$$

and

$$L_\nu(mz) = m_0 K_{\nu+2}(mz) - \frac{1}{m\beta} K_{\nu+1}(mz) .$$

The total energy is positive, which represents the expectation energy value of real fermions above the Dirac sea. It contains all the quantum corrections to the Boltzmann limit, which is the first term as $m = 1$.

The reliability of some function describing a physical quantity whose behaviour is subject to prediction by means of a grand canonical ensemble is measured by the mean-square deviation from the predicted value. In a mesoscopic system, the criterion for the ensemble to make an accurate prediction of the measurable value is significant. The physically measurable fluctuations provide measures of the uncertainty in the predictions. The variance of the total energy in the equilibrium system gives

$$\langle E^2\rangle - \langle E\rangle^2 = \frac{T}{\beta} C_L \tag{30}$$

where C_L is the heat capacity defined by (13), and for the free-fermion mesoscopic system in the weakly degenerate case we have

$$C_L = \frac{\xi K_{\mathrm{B}}\eta}{\pi} \sum_{m=1}^{\infty} (-1)^{m+1} \exp(m\beta\mu)\tilde{M}(mz) \tag{31}$$

where

$$\tilde{M}(mz) = M_0(mz) + \sum_{\ell=1}^{\infty}\sum_{\nu=0}^{\infty} P_{\nu\ell}(mz)M_\nu(mz) \cos\left(2\pi\ell\frac{\phi}{\phi_0}\right)$$

and

$$M_\nu(mz) = \mu\beta K_{\nu+1}(mz) - z(3 + \mu m\beta)K_{\nu+2}(mz) + mz^2 K_{\nu+3}(mz) .$$

Hence the statement about the behaviour of C_L in (31) is a measure of the actual physical fluctuation of the energy in the system. One visualizes that due to the coherence of the fermions, the energy fluctuation vibrating violently is a significant value which discribes the uncertainty in the mesoscopic system.

The weakly degenerate case is valid when $\exp[-\beta(m_0 - \mu)] \leqslant 1$; for this situation the particle density is higher or the temperature of the system somewhat lower than that of the Boltzmann statistical case. As the temperature increases, or the particle density decreases, the series of each of the above formulae converges very rapidly, due to $\exp[-\beta(m_0 - \mu)] \ll 1$. Thus, one regains the Boltzmann limit by retaining only the first term in each series of the formulae.

3.2. Non-relativistic limits

As $z \gg 1$, employing the asymptotic behaviour of the modified Bessel function $K_\nu(r)$, we can get the non-relativistic functions of the mesoscopic system directly from the explicit formulae as given above.

From equation (21) we derive the non-relativistic persistent current by employing the asymptotic behaviours of the modified Bessel function as $m_0 \gg K_{\mathrm{B}}T$ to give

$$I(\beta,\phi) = I_0 \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{\nu=0}^{\infty}(-1)^m \frac{\ell}{m^{3/2}} P_{\nu\ell}(mz)\exp(-m\lambda) \times \left[1 + \frac{4(\nu+1)^2-1}{8mz} + \mathrm{O}(z^{-2})\right]\sin\left(2\pi\ell\frac{\phi}{\phi_0}\right) \tag{32}$$

where

$$I_0 = \frac{qL\eta}{\beta\hbar\Lambda} \qquad \Lambda = \left(\frac{2\pi\hbar^2\beta}{m_0}\right)^{1/2} \qquad \lambda = \beta(m_0-\mu) > 0\,.$$

Arranging the series and taking the summation over the subscript ν we have

$$I(\beta,\phi) = 2I_0 \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m \frac{\ell}{m^{3/2}} \exp(-m\lambda - \eta_{m\ell})[1+\Delta_{m\ell}]\sin\left(2\pi\ell\frac{\phi}{\phi_0}\right) \tag{33}$$

where

$$\Delta_{m\ell} = \frac{1}{8\pi z}(4\eta_{m\ell}^2 - 12\eta_{m\ell} + 3) \qquad \eta_{m\ell} = \frac{(\xi\ell)^2}{2mz}\,.$$

The first term in the square bracket of (33) represents the non-relativistic current of the system, which is exactly the same equation as (A7) presented in [16] by considering the electron system, in which the electron charge $q = e$, and the spin number s is not reckoned. As the temperature is much greater than the characteristic temperature $\tilde{T} = \hbar^2/m_0 K_{\mathrm{B}} L^2$, the current decays rapidly in the exponential form with its magnitude fluctuating around the non-relativistic current as the temperature T increases. In the description of characteristic temperature, $\eta_{m\ell}$ can be rewritten as $\eta_{m\ell} = \ell^2 T/2m\tilde{T}$. The correction $\Delta_{m\ell}$ is a quadratic polynomial of $\eta_{m\ell}$, and it is zero at $a = (3-\sqrt{6})/2, b = (3+\sqrt{6})/2$. $\Delta_{m\ell}$ is positive for $a > \eta_{m\ell} > b$, and is negative for $a < \eta_{m\ell} < b$. For $\eta_{m\ell} = \frac{3}{2}$, $\Delta_{m\ell}$ reaches the lowest value of the correction curve to result in $\Delta_{m\ell} = -3/(4mz)$. Therefore, by the relativistic consideration the persistent current (33) is enhanced in the region $a > \eta_{m\ell} > b$, while it is weakened in the region $a < \eta_{m\ell} < b$. The relativistic correction exhibits descreteness as ladder to enhance and weaken the current. As $L \to \infty$, we see that equation (33) approaches zero; this confirms the fact that for normal metal persistent currents can only be observed in small closed system. As $T \gg \tilde{T}$, the expansion (33) decays very rapidly, and the first harmonic term ($m = 1$) gives a good approximation to the current.

Defining the function $f_\sigma^{(\ell)}(\lambda)$ by the following equation:

$$f_\sigma^{(\ell)}(\lambda) = \sum_{m=1}^{\infty}(-1)^m \frac{1}{m^\sigma}\exp(-m\lambda - a_\ell/m)$$

the persistent current (33) yields

$$I(\beta,\phi) = 2I_0 \sum_{\ell=1}^{\infty}\ell I_\ell(\lambda)\sin\left(2\pi\ell\frac{\phi}{\phi_0}\right) \tag{34}$$

where

$$I_\ell(\lambda) = f_{3/2}^{(\ell)}(\lambda) + \frac{1}{8z}[4a_\ell^2 f_{9/2}^{(\ell)}(\lambda) - 12a_\ell f_{7/2}^{(\ell)}(\lambda) + 3f_{5/2}^{(\ell)}(\lambda)] \qquad a_\ell = \ell^2 T/2\tilde{T}\,.$$

By this representation the persistent current is described completely by the function $f_\sigma^{(\ell)}(\lambda)$,and the usual non-relativistic situation merely relates to $f_{3/2}^{(\ell)}(\lambda)$.

The non-relativistic number density then obtained from (28) is that

$$n = -n_0\left[\tilde{n}_0(\lambda) + 2\sum_{\ell=1}^{\infty}\tilde{n}_\ell(\lambda)\cos\left(2\pi\ell\frac{\phi}{\phi_0}\right)\right] \tag{35}$$

where

$$\tilde{n}_\ell(\lambda) = f_{1/2}^{(\ell)}(\lambda) + \frac{1}{8z}[4a_\ell^2 f_{7/2}^{(\ell)}(\lambda) - 12a_\ell f_{5/2}^{(\ell)}(\lambda) + 3f_{3/2}^{(\ell)}(\lambda)]$$

and

$$n_0 = \frac{2\eta}{h}(2\pi m_0 K_{\rm B} T)^{1/2}\,.$$

The function $f_\sigma^{(\ell)}(\lambda)$ approaches zero when the circumference L approaches infinity for $\ell \neq 0$. Thus, the number density of macroscopic system is realized by

$$n = -n_0\left[f_{1/2}^{(0)}(\lambda) + \frac{3}{8z}f_{3/2}^{(0)}(\lambda)\right] \tag{36}$$

where the second term is the relativistc correction. The usual non-relativistic number density of this mesoscopic system is determined by $f_{1/2}^{(\ell)}(\lambda)$.

The non-relativistic total energy is found to be

$$E(\beta,\phi) = -E_0\left[\tilde{E}_0(\lambda) + 2\sum_{\ell=1}^{\infty}\tilde{E}_\ell(\lambda)\cos\left(2\pi\ell\frac{\phi}{\phi_0}\right)\right] \tag{37}$$

where

$$\tilde{E}_\ell(\lambda) = f_{1/2}^{(\ell)}(\lambda) + \frac{1}{8z}[4a_\ell^2 f_{7/2}^{(\ell)}(\lambda) - 20a_\ell f_{5/2}^{(\ell)}(\lambda) + 7f_{3/2}^{(\ell)}(\lambda)] \qquad E_0 = \frac{\xi m_0\eta}{(2\pi z)^{1/2}}\,.$$

The total energy density of the macroscopic system is

$$E(\beta)/L = -\frac{m_0^2\eta}{\hbar(2\pi z)^{1/2}}\left[f_{1/2}^{(0)}(\lambda) + \frac{7}{8z}f_{3/2}^{(0)}(\lambda)\right] \tag{38}$$

in which the second term is the relativistic correction to the lowest order of $1/z$. The usual non-relativistic energy density is associated to the function $f_{1/2}^{(\ell)}(\lambda)$, and the energy fluctuation defined by (30) is now measured by the non-relativistic heat capacity

$$C_L = C_0\left[\tilde{C}_0(\lambda) + 2\sum_{\ell=1}^{\infty}\tilde{C}_\ell(\lambda)\cos\left(2\pi\ell\frac{\phi}{\phi_0}\right)\right] \tag{39}$$

where

$$\tilde{C}_\ell(\lambda) = z^{3/2}(\mu - m_0)f_{-1/2}^{(\ell)}(\lambda) + \tfrac{1}{8}z^{1/2}g_{1\ell}(\lambda) + \tfrac{1}{128}z^{-1/2}g_{2\ell}(\lambda) + {\rm O}(z^{-3/2})$$

and

$$g_{1\ell}(\lambda) = 4a_\ell^2(\mu - m_0)f_{5/2}^{\ell}(\lambda) - 4a_\ell(5\mu - 7m_0)f_{3/2}^{(\ell)}(\lambda) + (7\mu - 11m_0)f_{1/2}^{(\ell)}(\lambda)$$

$$g_{2\ell}(\lambda) = 16a_\ell^4(\mu - m_0)f_{9/2}^{(\ell)}(\lambda) - 4a_\ell^3(56\mu - 57m_0)f_{7/2}^{(\ell)}(\lambda) + 2a_\ell^2(388\mu - 659m_0)f_{5/2}^{(\ell)}(\lambda)$$
$$-2a_\ell(324\mu - 779m_0)f_{3/2}^{(\ell)}(\lambda) + (57\mu - 225m_0)f_{1/2}^{(\ell)}(\lambda)\,.$$

By noting that for the weakly degenerate case $\mu \leqslant m_0$ and the first term in (39) plays a leading role in the function, one realizes the intrinsically positive character of the heat capacity $C_L \geqslant 0$. In the mesoscopic system the heat capacity is sensitively related to L, and the macroscopic heat capacity is smeard not only by the finite circumference but by the relativistic effect. The positive heat capacity is one of the thermodynamic stability conditions. The heat capacity of a macroscopic ring is given by taking the limit $L \to \infty$ in (39):

$$C_L = C_0\big[z^{3/2}(\mu - m_0) f^{(0)}_{-1/2}(\lambda) + \tfrac{1}{8} z^{1/2}(7\mu - 11m_0) f^{(0)}_{1/2}(\lambda) + \tfrac{1}{128} z^{-1/2}(57\mu - 225m_0) f^{(0)}_{1/2}(\lambda)\big] .$$

Therefore, in the weakly degenerate mesoscopic relativistic system, the physical quantities are sensitively associated with the dimension of the system and the temperature. The relativistic effect gives small corrections to these quantities by adding some exponential-decay-like terms. The fermions confined in a one-dimensional ring with circumference L comparable with phase-coherence length exhibit strong interferences. Relativistic statistical thermodynamic functions have complicated structures relating to the modified Bessel functions of the second kind, and they include all the relativistic corrections as well as the quantum properties.

4. The degenerate case

At very low temperature and high densities, the behaviour of the Fermi system is completely dominated by quantum statistical effects. Because the considerable overlap of wavefunctions is very large, the range of the chemical potential is positive, in fact here $\mu \geqslant m_0$. For this degenerate Fermi system, the fugacity expansions used in the weakly degenerate case cannot be employed as they stand. In order to calculate the grand partition function expressed generally in (16), we close the contour in the integral to the left with a semicircle of radius R in the anticlockwise direction, and then we deform the contour so as to avoid the branch cut along the negative real axis. There are no singularities within the contour, and the integrand vanishes on large radial circular arcs as $R \to \infty$. Thus the integral around the contour vanishes as a consequence of Chauchy's theorem. Let γ denote the portion of the contour from $-R$ on the x axis in the uper plane to $-r$, and a small circle of radius r around the origin in the clockwise direction, then the integral path runs from $-r$ to $-R$ on the x axis in the lower plane. Thus, equation (16) reduces to

$$\ln Z_{\mathrm{G}}(\beta, \phi) = -\frac{1}{2\mathrm{i}} \int_\gamma \frac{1}{t \sin(\pi t)} \exp(\beta\mu t) Z_{\mathrm{B}}(\beta t, \phi)\, \mathrm{d}t \qquad (40)$$

where the function $Z_{\mathrm{B}}(\beta t, \phi)$ is given by (19). The evaluation of (40) gives the logarithm of partition function. Note that in the caculation we should take the limit as $R \to \infty$, and $r \to 0$. The poles at negative integers on the cut will contribute only terms of exponential orders. Introducing the parameter $x = p_0/m_0$, and the notation $\tau = (x^2+1)^{1/2}$, where p_0 is the positive solution of $\mu = [p_0^2 + m_0^2]^{1/2}$ we have $\mu = \tau m_0$. The variable t is now changed to $t = u/\tau z$. We expect to carry out the calculations with expansion in $(1/\tau z)$. Dividing the integration into two parts, one is along the x-axis, the other is around the small circle, and then utilizing the relation among the modified Bessel functions

$$K_n[z \exp(\pm \mathrm{i}\pi)] = \exp(-\mathrm{i}n\pi) K_n(z) \mp \mathrm{i}\pi I_n(z) \qquad (41)$$

we reduce equation (40) to

$$\ln Z_{\mathrm{G}}(\beta,\phi) = \xi\eta\left[J_{\gamma 0}(\beta) + 2\sum_{\ell=1}^{\infty}\sum_{\nu=0}^{\infty}\tilde{P}_{\nu\ell}J_{\gamma\nu}(\beta)\cos\left(2\pi\ell\frac{\phi}{\phi_0}\right)\right] \tag{42}$$

where

$$\begin{aligned}\tilde{P}_{\nu\ell} &= (-1)^{\nu}\frac{1}{\nu!}\left(\frac{\xi\ell}{2}\right)^{2\nu}\\ J_{\gamma\nu} &= \lim_{r\to 0}(J_{1\nu} + J_{2\nu})\end{aligned} \tag{43}$$

and

$$\begin{aligned}J_{1\nu} &= (-1)^{\nu}\int_{r}^{\infty}\frac{\exp(-y)}{\sin(\pi y/\tau z)}I_{\nu+1}(y/\tau)\frac{\mathrm{d}y}{y^{\nu+1}}\\ J_{2\nu} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\exp[u(\theta)]}{\sin[\pi u(\theta)/\tau z]}K_{\nu+1}\left[\frac{u(\theta)}{\tau}\right]u^{-\nu}(\theta)\,\mathrm{d}\theta\,.\end{aligned}$$

By $u(\theta)$ we mean that $u(\theta) = r\exp(\mathrm{i}\theta)$. Here $I_{\nu}(z)$ is the modified Bessel function of the first kind, and when ν is a positive integer, it has the power expansion of z. The function is also defined for ν to be complex. In particular, $I_{\nu}(z)$ is real and positive when $\nu > -1, z > 0$, and is an entire function of z when ν is one of the positive or negative integers. In equation (42), although integral $J_{1\nu}$ converges at the uper infinite limit, there exists a logarithmic divergence at the lower limit as $r \to 0$, which diverges at the origin as

$$(-1)^{\nu+1}\frac{1}{(\nu+1)!}\frac{\tau z}{\pi}\left(\frac{1}{2\tau}\right)^{\nu+1}\ln r\,. \tag{44}$$

The integral $J_{2\nu}$ also contains a logarithmic divergence at the same point with exactly the same form as that of (44) but an opposite sign. However, this singularity can be isolated by expanding the sine function in this integral. As a result, we realize that the logarithmic divergences precisely are cancelled from each other in (42). Therefore, the logarithm of partition function is convergent at the origin. Integrate once by parts in the integral $J_{1\nu}$, and note that the recurrence relations for $I_{\nu}(z)$ allow us to get the differential relation

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[\mathrm{e}^{-y}I_{\nu+1}(y/\tau)\frac{1}{y^{\nu+1}}\right] = \mathrm{e}^{-y}\left[\frac{1}{\tau}I_{\nu+2}(y/\tau) - I_{\nu+1}(y/\tau)\right]\frac{1}{y^{\nu+1}} \tag{45}$$

then use the expansion formulae of modified Bessel functions $K_{\nu}(z)$ and $I_{\nu}(z)$ to integrate (43) term by term, one arrives at

$$J_{\gamma\nu}(\beta) = (-1)^{\nu}\left[\frac{z}{2\pi}J_{\nu}^{(1)}(\tau) + \frac{\pi}{24z\tau^2}J_{\nu}^{(2)}(\tau) + \frac{7}{1440\tau}(\frac{\pi}{\tau z})^3 J_{\nu}^{(3)}(\tau)\right] \tag{46}$$

where

$$\begin{aligned}J_{\nu}^{(1)}(\tau) = {}& \frac{1}{(\nu+1)!}\left[\ln\left(\frac{1}{2\tau}\right) + \psi(1)\right] + \frac{1}{2}\sum_{k=0}^{\nu}(-1)^k\frac{k!}{(\nu-k)![2(k+1)]!}(2\tau)^{2(k+1)}\\ &+\sum_{k=0}^{\infty}\frac{(2k+1)!}{(k+\nu+2)!(k+1)!}\left(\frac{1}{2\tau}\right)^{2(k+1)}\\ J_{\nu}^{(2)}(\tau) = {}& \sum_{k=0}^{\nu}(-1)^k\frac{k!}{(\nu-k)!(2k)!}\left(\frac{1}{2\tau}\right)^{-2(k+1)} + 2\sum_{k=0}^{\infty}\frac{(2k+1)!}{k!(k+\nu+1)!}\left(\frac{1}{2\tau}\right)^{2k}\\ J_{\nu}^{(3)}(\tau) = {}& \sum_{k=0}^{\nu-1}(-1)^{k+1}\frac{(k+1)!}{(2k)!(\nu-k-1)!}(2\tau)^{2(k+2)} + 2\sum_{k=0}^{\infty}\frac{(2k+3)!}{k!(k+\nu+1)!}\left(\frac{1}{2\tau}\right)^{2k}\end{aligned}$$

and $\psi(1) = -0.577\ 216\ldots$, the negative Euler–Mascheroni constant. Therefore, the logarithm of partition function for the degenerate relativistic quantum ideal fermion system is given as the infinite series by substituting equation (46) into (42). Our result of the degenerate system is related to the hypergeometric series with one and two variables [39]:

$$F(\alpha_1, \alpha_2; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n}{n!(\gamma)_n} x^n$$

$$\Phi_2(\alpha_1, \alpha_2; \gamma; x, y) = \sum_{k=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k}{k!\nu!(\gamma)_{k+\nu}} x^k y^\nu$$

and

$$\Theta_1(\alpha_1, \alpha_2; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha_1)_m}{k!m!(\alpha_2)_m (\gamma)_{k+m}} x^m y^k$$

where x, y are variables, and $(\alpha)_n$ are defined by

$$(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)$$

i.e.

$$(\alpha)_0 = 1, (\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1) \qquad n = 1, 2, \ldots .$$

Thus, we obatin the logarithm of partition function as

$$\ln Z_{\mathrm{G}}(\beta, \phi) = \xi\eta \left[J_0(\beta) + \sum_{\ell=1}^{\infty} \tilde{J}_\ell(\beta) \cos\left(2\pi\ell \frac{\phi}{\phi_0}\right) \right] \tag{47}$$

where

$$J_0(\beta) = \frac{z}{2\pi} \left[\ln\left(\frac{1}{2\tau}\right) + \psi(1) + \tau^2 + \frac{1}{8} \int_0^{1/\tau^2} F(1, 3/2; 3; x)\, \mathrm{d}x \right]$$
$$+ \frac{\pi}{12z} \left[\frac{1}{\tau^2} F(1, 3/2; 2; 1/\tau^2) + 2 \right] + \frac{7}{120\tau} \left(\frac{\pi}{\tau z}\right)^3 F(2, 5/2; 2; 1/\tau^2)$$

and

$$\tilde{J}_\ell(\beta) = \frac{z}{\pi} J_\ell^{(1)}(\tau) + \frac{\pi}{12z\tau^2} J_\ell^{(2)}(\tau) + \frac{7}{720\tau} \left(\frac{\pi}{\tau z}\right)^3 J_\ell^{(3)}(\tau)$$

with

$$J_\ell^{(1)}(\tau) = \frac{2}{\xi\ell} \left[\ln\left(\frac{1}{2\tau}\right) + \psi(1) \right] I_1(\xi\ell) - \left(\frac{2}{\xi\ell}\right)^2 \int_0^{x_\ell} \Theta_1(1, 3/2; 1; x, y_\ell)\, \mathrm{d}x$$
$$+ \frac{1}{2^3} \int_0^{1/\tau^2} \Phi_2(3/2, 1; x, y_\ell)\, \mathrm{d}x$$

$$J_\ell^{(2)}(\tau) = (2\tau)^2 \Theta_1(1, 1/2; 1; x_\ell, y_\ell) + 2\Phi_2(3/2, 1; 2; 1/\tau^2, y_\ell)$$

$$J_\ell^{(3)}(\tau) = 12\Phi_2(5/2, 2; 2; 1/\tau^2, y_\ell) - \tau^4 (2\xi\ell)^2 \Theta_1(2, 1/2; 2; x_\ell, y_\ell)$$

$$x_\ell = -\left(\frac{\xi\ell\tau}{2}\right)^2 \qquad y_\ell = \left(\frac{\xi\ell}{2}\right)^2 .$$

Equation (47) converges under the condition $\mu > m_0$, by which we can derive the thermodynamic functions directly. The persistent current now expressed by the formula

$$I(\beta, \phi) = -\tilde{I}_0 \sum_{\ell=0}^{\infty} \ell \tilde{J}_\ell(\beta) \sin\left(2\pi\ell \frac{\phi}{\phi_0}\right) \tag{48}$$

where

$$\tilde{I}_0 = \frac{\xi\eta q}{\beta\hbar}$$

and $\tilde{J}_\ell(\beta)$ is given by (47). The current decays with temperature in power form, but is a constant at zero temperature. From equations (10) and (47), one immediately obtains the particle number density

$$n = n_0' \left[\tilde{n}_0 + \sum_{\ell=0}^{\infty} \tilde{n}_\ell \cos\left(2\pi\ell\frac{\phi}{\phi_0}\right)\right] \tag{49}$$

where

$$\tilde{n}_0 = \frac{z}{\pi}\left[(2\tau)^2 - 1 - \frac{1}{8}\int_0^{1/\tau^2} F(1, 3/2; 3; x)\,\mathrm{d}x\right] - \frac{\pi}{3z\tau^2} F(3/2, 1; 1; 1/\tau^2)$$

$$\tilde{n}_\ell = \frac{z}{\pi}\left[-\frac{2}{\xi\ell} I_1(\xi\ell) + \frac{1}{2}(2\tau)^2\Theta_1(1, 3/2; 1; x_\ell, y_\ell) - \frac{1}{8}\int_0^{1/\tau^2} \Phi_2(1, 3/2; 3; x, y_\ell)\,\mathrm{d}x\right]$$
$$+ \frac{\pi}{3z\tau}\left[2\tau\Theta_1(1, 1/2; 1; x_\ell, y_\ell) - \frac{1}{\tau}\Phi_2(2, 3/2; 2; 1/\tau^2, y_\ell)\right]$$

and

$$n_0' = \frac{m_0\eta}{2z\tau\hbar}.$$

The number density is not independent of dimension, as the case in a macroscopic system, but depends on the circumference. The total energy is thus found by substituting equations (47), (49) into (11):

$$E(\beta, \phi) = \xi m_0 \eta \left[\tilde{E}_0(\beta) + 2\sum_{\ell=0}^{\infty} \tilde{E}_\ell(\beta) \cos\left(2\pi\ell\frac{\phi}{\phi_0}\right)\right] \tag{50}$$

where

$$\tilde{E}_0(\beta) = \frac{1}{2\pi}\left[-d + (\tau/2)^2 \int_0^{1/\tau^2} F(3/2, 1; 3; x)\,\mathrm{d}x\right]$$
$$+ \frac{\pi}{6z^2}\left[1 - \frac{1}{2\tau^2}\left(2x_1\frac{\partial}{\partial x_1} + 1\right) F(3/2, 1; 2; x_1)\right]$$

$$\tilde{E}_\ell(\beta) = \frac{1}{2\pi}\left[-\frac{2d}{\xi\ell} I_1(\xi\ell) + \frac{2\tau^2}{x_\ell}\int_0^{x_\ell} \Theta_1(1, 1/2; 1; x, y_\ell)\,\mathrm{d}x\right.$$
$$\left. - \frac{1}{4}\int_0^{1/\tau^2} \Phi_2(3/2, 1; 3; x, y_\ell)\,\mathrm{d}x\right]$$
$$+ \frac{\pi}{6z^2}\left[\left(2x_\ell\frac{\partial}{\partial x_\ell} + 1\right)\Theta_1(1, 1/2; 1; x_\ell, y_\ell)\right.$$

$$-\frac{1}{2\tau^2}\left(2x_1\frac{\partial}{\partial x_1}+1\right)\Phi_2(3/2,1;2;x_1,y_\ell)\Bigg]$$

and

$$d=1+\psi(1)+\ln\left(\frac{1}{2\tau}\right) \qquad x_1=1/\tau^2\,.$$

The energy fluctuation defined by equation (30) is now measured in this degenerate case by the heat capacity for constant L, which can be drawn from (13). One sees that the heat capacity is positive, and it vanishes as the temperature approaches zero. The fluctuation of energy sensitively depends on th circumference of the ring. From equations (14) and (47) one can obtain the pressure of the Fermi gas, and one finds that the pressure is not zero even at the absolute zero of temperature. The Fermi gas must be confined owing to the pressure in the ground state, which is a manifestation of Pauli's principle.

The non-relativistic thermodynamic functions can be computed by letting the quantity x in $\tau=(1+x^2)^{1/2}$ to be small enough, i.e. $x\ll 1$, in the corresponding formulae for the degenerate situation. In our calculations we have included the rest energy in the Hamiltonian of the ideal free-fermion system, not merely presenting the kinetic energy. So that in the non-relativistic correspondents the contributions of the rest energy to the functions are also included. As $\mu\gg K_\mathrm{B}T$, the expansions of degenerate thermodynamic functions with the power form $1/(z\tau)$ converge rapidly, and they give good approximations.

5. Discussion

We have examined the relativistic quantum behaviours of ideal many-fermion system confined in a small one-dimensional ring with circumference L threaded by an external magnetic flux. We study the system starting from inverse Mellin transform to obtain the logarithms of partition functions both in the weakly degenerate and degenerate cases, from which we havefound persistent currents, number densities, total energies and heat capacity. These results contain relativistic and quantum corrections of Boltzmann statistical system. These quantities are periodic functions of magnetic flux ϕ with period ϕ_0, and they are sensitive to the circumference L. The mesoscopic system presents discrete energies evidently arising from the finite special confinement of the sample. The particles interfere strongly as L is of the order of phase coherence length, and fluctuations of observables are of significance. Each of the measured quantities is related to the modified Bessel functions of the second kind, which is a special property of relativistic quantum statistical many-body system. However, in the mesoscopic ring, they are concerned with $K_\nu(z),\ \nu=0,1,\ldots,$ which is much complicated than that of a macroscopic system where they are only related to $K_1(z)$. For a d-dimensional macroscopic system, the logarithm of its partition function are functions of $K_{(d+1)/2}(z),\ d=1,2,3$. Therefore, the relativistic systems contain much more subtle information than that of non-relativistic systems, and we also have very many mathematical applications in them.

For the weakly degenerate case, where $\mu\leqslant m_0$, we obtain the persistent current (21), number density (28), total energy (29), and energy fluctuation measured by the heat capacity (31). These quanties decay rapidily when $\mu\ll m_0$, and the leading terms in each of these expansions retain the corresponding Boltzmann statistical results. The non-relativistic relevant behaviours are derived by letting $m_0\gg K_\mathrm{B}T$ (for $C=1$), and we found these quantities are fluctuating around the usual non-relativistic terms. This fact represents that the relativistic consideration of the system gives pictures of enhancement and reduction of the corresponding physical measured values. Specifically, if we take the system of fermions to be electron system, we can have the relativistic picture of electron transport with relativistic corrections. For the weakly degenerate electron situation, where the fugacity expansion is valid, this corresponds to the case of electrons transporting in insulation materials. The persistent current circulating around the ring increases evidently with temperature, because the filled fermions gain enough kinetic energies to participate the conduction. The current oscillates periodically with ϕ as a sine function, and the non-relativistic persistent electric current limit is coincident with that obtained in [16]. In order to keep the equilibrium state, the variation of the magnetic flux must be sufficiently slow. The number density, total energy, heat capacity are vibratory functions of cosine with period ϕ_0. The energy fluctuation measured by the heat capacity gives the stability condition of the equilibrium state. All these thermodynamic quantities decline in magnitude exponentially with temperaure and the circumference. They exhibit explicitly relativistic corrections in the magnitudes.

For the degenerate case, where $\mu > m_0$, we have evaluated the relevant observables. The main results are equations (47), (48), (49) and (50). These thermodynamic observables are characterized by temperature, chemical potential, circumference and magnetic flux. They are related to the generalized hypergeometric functions with two variables. Therefore we find an application of these specific functions in our mesoscopic system. As $\mu > m_0 \gg K_B T$, these expansions converge quickly with temperature declining in the power form. However, they are not zero even at the absolute zero temperature, but oscillate periodically with ϕ, and they are much sophisticated than the weakly degenerate circumference. All the discontinuities of these functions at absolute zero temperature are smeared to round in small temperature. The declines in magnitudes of these observables are not faster than those of the weakly degenerate case. Non-relativistic limits can also be reduced from obtained results by considering the velocities of fermions to be very small compared with that of light. Defining the Fermi energy by the kinetic energy formula, one immediately finds corresponding formulae for the zero temperature. One can draw from above results that the pressure is not zero even at zero temperature, which is the natural behaviour of Pauli's principle.

As the velocities of particles are large enough, it is necessary to deal with the problems by means of special relativistic theory. This can provide us some usefull subtle information on the particle systems, and it can also exhibits some mathematical structures for investigation. However, employing relativistic quantum statistical mechanics to dispose of many-body problems including particle–particle interactions is a very complicated thing, in which one has to tackle the difficult task of creation and destruction. Therefore, although the free-fermion model is unrealistic, it can give us some main configuration on the physical system.

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